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ON EDGE-HAMILTONIAN PROPERTY OF CAYLEY GRAPHS

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Let G be a group generated by X. A *Cayley graph over* G is defined as a graph $G(X)$ whose vertex set is G and whose edge set consists of all unordered pairs $[a, b]$ with $a, b \in G$ and $a^{-1}b \in X \cup X^{-1}$, where X^{-1} denotes the set $\{x^{-1} \mid x \in X\}$. When X is a minimal generating set or each element of X is of even order, it can be shown that $G(X)$ is Hamiltonian iff it is edge-Hamiltonian. Hence every Cayley graph of order a power of 2 is edge-Hamiltonian.

1. Introduction

Throughout this paper, we shall only consider finite undirected simple graphs. For each such graph Γ, we shall denote the vertex set of Γ by $V(\Gamma)$ and the edge set by $E(\Gamma)$. The letter G will always denote a finite group generated by a set X where the identity ι of G is not in X. A *Cayley graph over* G is defined as a graph $G(X)$ whose vertex set is G and whose edge set consists of all unordered pairs $[a, b]$ with $a, b \in G$ and $a^{-1}b \in X \cup X^{-1}$ ($X^{-1} = \{x^{-1} \mid x \in X\}$). In other words, two vertices a, b of $V(G)$ are adjacent iff $ax = b$ or $ax^{-1} = b$ for some x in X. We shall call the corresponding edge $[a, b]$ an x-*edge* and a, b the *ends* of $[a, b]$. It is easy to see that each element a of G gives rise to an automorphism Θ_a of $G(X)$ where $\Theta_a : x \to ax$. Further, the mapping $a \to \Theta_a$ is an embedding of G into the automorphism group of $G(X)$. Because of this, it is clear that the graph $G(X)$ is vertex-transitive in the sense that for any two vertices a, b of $G(X)$, there exists an automorphism (namely, Θ_c where $c = ba^{-1}$) of $G(X)$ which maps a to b. In fact, the converse is almost true, for there are only four known examples of connected vertex-transitive graphs which are not Cayley graphs.

Among the many conjectures which intrigue combinatorists nowadays, the following conjecture of Lovasz is undoubtedly one that attracts much attention.

Conjecture 1. [Lovasz] Every connected vertex-transitive graph has a Hamilton path.

As Cayley graphs form a special class of vertex-transitive graphs, the above conjecture naturally leads to the following more specific conjecture:

Conjecture 2. Every Cayley graph has a Hamilton cycle.

Though much effort has been made in order to verify the conjecture, yet up to date, it is known that the conjecture is only true for Cayley graphs over some very special classes of groups. For instance, it is fairly easy to show that Cayley graphs over Abelian groups are always Hamiltonian. However, for Cayley graphs over non-Abelian groups, the conjecture is far from being solved. Chen and Quimpo prove in [2] that Cayley graphs over groups of order pq, where p, q are primes, are Hamiltonian and in [3] that Cayley graphs over Hamiltonian groups (i.e. non Abelian groups in which every subgroup is normal) are Hamiltonian. The best known result so far is perhaps the one proved by Keating and Witte in [4], which states that if G is a group whose commutator subgroup is a cyclic group of prime power order, then any Cayley graph over G is Hamiltonian. Witte [5] also showed recently that every Cayley graph of prime power order is Hamiltonian. Furthermore, for most of these known results, it can be shown that the corresponding Hamiltonian Cayley graphs are in fact *edge-Hamiltonian,* in the sense that every edge lies on a Hamilton cycle. This in turn leads to another interesting conjecture:

Conjecture 3. Every Hamiltonian Cayley graph is edge-Hamiltonian.

The main objective of this paper is to give a partial solution to this conjecture. Among others, we shall show that when each element of X is of even order, $G(X)$ is Hamiltonian iff it is edge-Hamiltonian. Hence Cayley graph over a group of order a power of 2 is always edge-Hamiltonian.

2. Basic lemmas

The following lemmas will be useful in the sequel.

Lemma 1. *A Cayley graph $G(X)$ is edge-Hamiltonian iff for each x in X, $G(X)$ contains a Hamilton cycle containing an x-edge.*

Proof. Apparently if $G(X)$ is Hamiltonian, then the given condition holds. Conversely, assume that for each x in X, $G(X)$ contains a Hamilton cycle with an x-edge. To prove that $G(X)$ is edge-Hamiltonian, let $e=[a, b]$ be any edge of $G(X)$ with $a^{-1}b=x\in X\cup X^{-1}$. Without loss of generality, we may assume that $x\in X$. By the given condition, G has a Hamilton cycle C containing an edge $e'=[c, d]$ with $c^{-1}d=x$. Now consider the mapping $\Theta_g: y\rightarrow gy$ where $g=ac^{-1}$. Then Θ_g is an automorphism of $G(X)$ and so $\Theta_g(C)$ is also a Hamilton cycle of $G(X)$. However, $\Theta_g(c)=gc=a$ and $\Theta_g(d)=gd=gcc^{-1}d=gcx=ac^{-1}ca^{-1}b=b$. Hence the edge $[a, b]$ lies on the Hamilton cycle $\Theta_g(C)$ which proves that $G(X)$ is edge-Hamiltonian, as required. □

Lemma 2. *Let C be a Hamilton cycle of a Cayley graph $G(X)$. Let $Y = \{a^{-1}b \mid [a, b] \in E(C)\}$. Then Y generates G.*

Proof. Let the Hamilton cycle C be $\langle v_0, v_1, \ldots, v_n \rangle$, where n is the order of G and $v_0 = v_n = \iota$, the identity of G. Let $v_i^{-1}v_{i+1} = x_i$ where $i = 0, 1, \ldots, n-1$. Then the set $Y' = \{x_i \mid i = 0, 1, \ldots, n-1\}$ is contained by Y. Moreover, as each v_i is the product of $x_0, x_1, \ldots, x_{i-1}$, we see that Y' generates G and so Y also generates G, as required. □

Let Γ be a cubic graph. By a *perfect matching partition* (*PMP*) of Γ, we mean an unordered triple $\bar{P} = [P_1, P_2, P_3]$ where the P_i's are edge-disjoint perfect matchings of Γ. Apparently, they form a partition of $E(\Gamma)$ into perfect matchings. Moreover, by an *even cycle partition* (*ECP*) of Γ, we mean a set $\bar{C} = \{C_1, \ldots, C_k\}$ of vertex-disjoint cycles of Γ whose union is $V(\Gamma)$. Hence $\bar{C}$ is a partition of $V(\Gamma)$ into even cycles. It is easy to see that each (PMP) $\bar{P} = [P_1, P_2, P_3]$ gives rise to exactly three (ECP)'s, namely, $\bar{C}_1 = P_1 \cup P_2$, $\bar{C}_2 = P_2 \cup P_3$ and $\bar{C}_3 = P_3 \cup P_1$. On the other hand, for each (ECP) $\bar{C} = \{C_1, \ldots, C_k\}$ there are exactly 2^{k-1} (PMP)'s which give rise to $\bar{C}$, because each even cycle has two perfect matchings and the set of all edges not in $C_1, \ldots, C_k$ also form a perfect matching of Γ. With this in mind, we can now establish the following:

Lemma 3. *Every edge in a cubic graph lies on an even number of Hamilton cycles.*

Proof. Let Γ be any cubic graph and e an edge of Γ. Consider the following table each row of which corresponds to a (PMP) and each column of which corresponds to an (ECP). As each (PMP) $\bar{P} = [P_1, P_2, P_3]$ gives rise to three (ECP)'s $\bar{C}_1 = P_1 \cup P_2$, $\bar{C}_2 = P_2 \cup P_3$ and $\bar{C}_3 = P_3 \cup P_1$, we shall complete the table by filling in the three (ECP)'s $P_1 \cup P_2$, $P_2 \cup P_3$, $P_3 \cup P_1$ in the row corresponding to $\bar{P} = [P_1, P_2, P_3]$ and in the columns corresponding to $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$ respectively. All other entries in this row will be filled up by empty sets.

(PMP) \ (ECP)	$\cdots$	$\bar{C}_1$	$\cdots$	$\bar{C}_2$	$\cdots$	$\bar{C}_3$	$\cdots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$	
$\bar{P}$	$\cdots$	$P_1 \cup P_2$	$\cdots$	$P_2 \cup P_3$	$\cdots$	$P_3 \cup P_1$	$\cdots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$	

Now we shall count the number N of occurrences of the edge e in the whole table. Evidently, each row contains the edge e exactly twice. Hence, $N = 2r$ where r is the number of rows, or the number of (PMP)'s of Γ. On the other hand, if we count over a column corresponding to an (ECP) $\bar{C} = \{C_1, \ldots, C_k\}$, as this (ECP) occurs at exactly 2^{k-1} locations in the column, the number of occurrences of e in the column is either 0 or 2^{k-1} which is always even for $k>1$ and is equal to 1 for $k=1$, in which case $\bar{C}$ consists of exactly one Hamilton cycle. Hence the number N of occurrences of e in the table is equal to the sum of the number n_1 of Hamilton cycles containing e and an even number n_2. That is, $N=n_1+n_2=2r$. Therefore n_1 must be even, as required. □

Lemma 4. *Every Hamiltonian cubic graph contains at least* 3 *Hamilton cycles.*

Proof. Let Γ be a Hamiltonian cubic graph. By Lemma 3, G contains at least two Hamilton cycles C_1 and C_2, say. There must exist an edge e of Γ which is in C_1 but not in C_2. Again, by Lemma 3, there is a Hamilton cycle C_3 other than C_1 that also contains e. Evidently C_3 is different from C_2 and so we have at least three Hamilton cycles C_1, C_2 and C_3 in Γ. □

To end this section, we would like to raise the following question:

Problem. Does every regular Hamiltonian graph other than a cycle contains more than one Hamilton cycles? In particular, does every 4-regular Hamiltonian graph contain more than one Hamilton cycles?

3. Edge Hamiltonian property of Cayley graphs

With the basic lemmas established in the previous section, we are now in a position to prove the following theorems.

Theorem 5. *Let X be a minimal generating set of the group G. Then $G(X)$ is Hamiltonian iff it is edge-Hamiltonian.*

Proof. The sufficiency is clear. To prove the necessity, assume that $G(X)$ is Hamiltonian. Let C be any Hamilton cycle of G. It follows from Lemma 2 that the set $A=\{a^{-1}b \mid [a, b]\in E(C)\}$ generates G. As this is a subset of $X\cup X^{-1}$, by minimality of X, we must have $X\subset A$. Hence for each x in X, there exists an edge $[a, b]$ of C with $a^{-1}b=x$. Thus C contains an x-edge. Hence by Lemma 1, $G(X)$ is edge-Hamiltonian. □

Theorem 6. *Let $G(X)$ by a Cayley graph where each element of X is of even order. Then $G(X)$ is Hamiltonian iff $G(X)$ is edge-Hamiltonian.*

Proof. Again the sufficiently is clear. To prove the necessity, let x be any element of X. We shall show that $G(X)$ contains a Hamilton cycle with an x-edge. Let C be any Hamilton cycle of $G(X)$. If C contains an x-edge, then we are through. Assume therefore that C does not contain an x-edge.

If the order of x is 2, then C together with all the x-edges of $G(X)$ form a cubic Hamiltonian graph Γ in which all the x-edges are 'chords' of C. By Lemma 4, Γ must contain a Hamilton cycle C' other than C. Thus, Γ (which is also a Hamilton cycle of $G(X)$) must contain a chord which is an x-edge that we require.

Finally, consider the case when the order of x is $2k$ where k is an integer greater than 1. In this case, the set of all x-edges of $G(X)$ forms an (ECP) $\bar{D}$ of $G(X)$. The Hamilton cycle C together with $\bar{D}$ gives rise to a subgraph Ω of $G(X)$ which is a 4-regular Hamiltonian graph. As every even cycle contains a perfect matching, Ω contains a perfect matching P whose edges are chosen from cycles of $\bar{D}$. Hence C together with P form a subgraph Ω' of Ω which is a cubic Hamiltonian graph. Now as in the first part of the proof, Ω' contains another Hamilton cycle with an x-edge, which is also a Hamilton cycle of $G(X)$. As the element x of X is arbitrarily chosen, the proof that $G(X)$ is edge-Hamiltonian follows from Lemma 1. □

Combining this and the fact that Cayley graphs of prime power order are Hamiltonian (Witte [5]), we have the following immediate consequence:

Theorem 7. *Every Cayley graph of order* 2^k $(k = 2, 3, \ldots)$ *is edge-Hamiltonian.*

To end this paper, we wish to point out that from the proof of Theorem 6, it is clear that if the question to the problem raised at the end of the previous section is in the affirmative, then Conjecture 3 will also be established.

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